Virtual Group Knowledge and Group Belief in Topological Evidence Models

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Based on joint work with
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A topological account of knowledge

Why use topological semantics for epistemic logic?

- Richer, **evidence-based** notions of knowledge and belief.

We represent **evidence** topologically:

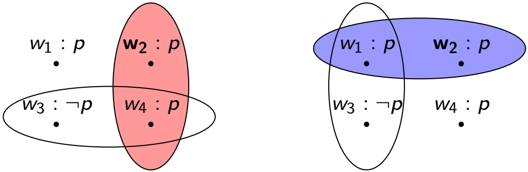
Hard evidence	Partition cells
Soft evidence	Open sets
Justifications	Dense open sets

Justifications are *consistent with all other evidence*.

On **topological evidence models** (topo-e-models),

- belief amounts to having a **supporting justification**,
- knowledge is interpreted as **correctly justified belief**,
- and knowledge is **fallible**!

Did Charles kill Daisy? Spoiler: yes! (w_2)



Alice's evidence Bob's evidence

- Intent to kill: $I := \{w_2, w_4\}$
- Caught in the act: $C = \{w_1, w_2\}$
- Guilty: $[\![p]\!] = I \cup C$
- **Not** guilty: $X \setminus [\![p]\!]$
- Jury foreperson Alice's evidence: $I, \neg C$
- Detective Bob's evidence: $C, \neg I$

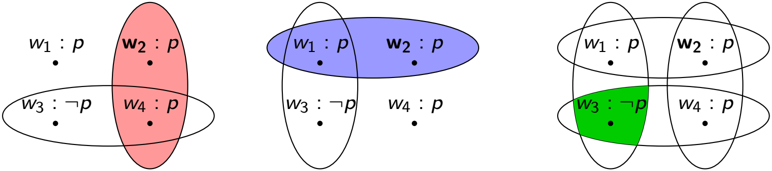
Would Alice and Bob as a group agree that Charles is guilty?

The goal: interpreting (distributed) knowledge of a group on multi-agent topo-e-models.

The approach: pooling together individual evidence.

The problem: a group may know (believe) less than (*any of*) its members!

Group knowledge: the "problem" of non-Monotonicity



Alice's evidence — Bob's evidence — The group's evidence

- Intent to kill: $I := \{w_2, w_4\}$
- Caught in the act: $C = \{w_1, w_2\}$
- Guilty: $[\![p]\!] = I \cup C$
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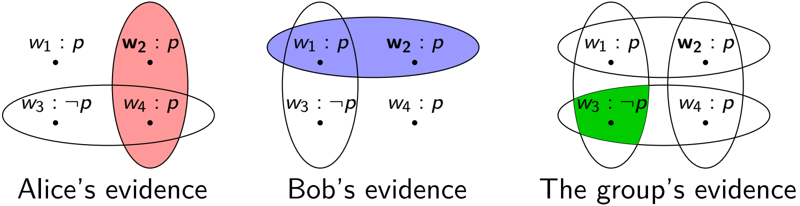

We have $\boldsymbol{w_2} \in \boldsymbol{K_a}(\boldsymbol{P}) \cap \boldsymbol{K_b}(\boldsymbol{P})$, but $\boldsymbol{w_2} \notin \boldsymbol{B_{\{a,b\}}}(\boldsymbol{P})$.

Charles is now considered **not guilty**... due to reasonable doubt!

This is a **feature**, not a **bug**!

Fallible knowledge must violate Group Monotonicity.

- Dynamics of belief must be non-monotonic;
- Realistic group knowledge should be "realizable";
- Our group knowledge is **not** traditional distributed knowledge!

Virtual group knowledge and virtual group belief describe epistemic potential of the group.

- Virtual group knowledge pre-encodes individual knowledge after sharing.

Definition 1 (Multi-Agent Topo-E-Model)

A ***multi-agent topo-e-model*** *for a group A of agents is a tuple*

$$(X, \Pi_i, \tau_i, [\![\cdot]\!])_{i \in A}$$

such that

1. X *is the state space*
2. *For each agent $i \in A$:*
   - $\Pi_i \subseteq \mathcal{P}(X)$ *is a* ***partition*** *of X* ***(Hard evidence)***
   - $\tau_i \subseteq \mathcal{P}(X)$ *is a* ***topology*** *on X* ***(Soft evidence)***
   - $\Pi_i \subseteq \tau_i$ ***(Hard evidence is evidence)***
3. $[\![\cdot]\!] : X \to \mathcal{P}(\mathrm{Prop})$ *is a valuation.*

Semantic operators:

Infallible knowledge	$[\forall]_i(P) := \{x \in X \mid \Pi_i(x) \subseteq P\}$
Factive evidence	$\Box_i(P) := \{x \in X \mid \exists U \in \tau_i : x \in U \subseteq P\}$ $= Int_i(P)$
Justified belief	$B_i(P) := \{x \in X \mid \Pi_i(x) \subseteq Cl_i(Int_i(P))\}$
Fallible knowledge	$K_i(P) := \{x \in X \mid \exists U \in \tau_i : x \in U \subseteq P$ and $\Pi_i(x) \subseteq Cl_i(U)\}$ $= \boldsymbol{Int_{\tau_i^{dense}}(P)}$

Useful equations:

- $B_i(P) = [\forall]_i \Diamond_i \Box_i(P)$
- $K_i(P) = \Box_i(P) \cap B_i(P)$

Definition 2 (Join Partition and Join Topology)

*Given agents I with individual partitions $(\Pi_i)_{i \in I}$ and topologies $(\tau_i)_{i \in I}$, the **join partition** for the group I is*

$$\Pi_I := \bigvee_{i \in I} \Pi_i = \{\Pi_I(x) \mid x \in X\} \quad \textit{where} \quad \Pi_I(x) := \bigcap_{i \in I} \Pi_i(x)$$

*and the **join topology** is the topology*

$$\tau_I = \bigvee_{i \in I} \tau_i \quad \textit{generated by the union} \quad \bigcup_{i \in I} \tau_i.$$

Group knowledge is **interior in the dense open join topology**.

Evidence-sharing dynamics

$\mathbf{share_I}$: the (semi-public) action of **Sharing (all evidence) within group $I \subseteq A$**.

$$\mathfrak{M} = (X, \Pi_i, \tau_i, [\![\cdot]\!]) \mapsto \mathfrak{M}(\mathsf{share}_I) := (X, \Pi(\mathbf{share}_I), \tau(\mathbf{share}_I), [\![\cdot]\!])$$

where

$$\begin{array}{llllll} \tau_i(\mathsf{share}_I) & = \tau_I, & \Pi_i(\mathsf{share}_I) & = \Pi_I & & (\text{for "insiders" } i \in I), \\ \tau_j(\mathsf{share}_I) & = \tau_j & \Pi_j(\mathsf{share}_I), & = \Pi_j & & (\text{for "outsiders" } j \notin I), \end{array}$$

Virtual group knowledge pre-encodes individual knowledge after sharing:

Proposition 1

For every proposition $P \subseteq X$:

$$\begin{array}{llll} [\forall]_i^{\mathfrak{M}(\mathsf{share}_I)}(P) & = [\forall]_I^{\mathfrak{M}}(P), & \Box_i^{\mathfrak{M}(\mathsf{share}_I)}(P) & = \Box_I^{\mathfrak{M}}(P), \\ B_i^{\mathfrak{M}(\mathsf{share}_I)}(P) & = B_I^{\mathfrak{M}}(P), & K_i^{\mathfrak{M}(\mathsf{share}_I)}(P) & = K_I^{\mathfrak{M}}(P). \end{array}$$

The language $\mathcal{L}_{\Box[\forall]_I}$ of **group evidence** is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \boldsymbol{\Box_I \varphi} \mid \boldsymbol{[\forall]_I \varphi}$$

and the language $\mathcal{L}_{KB_I}$ of **group knowledge and belief** is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \boldsymbol{B_I \varphi} \mid \boldsymbol{K_I \varphi}$$

where $p \in \text{Prop}$ and I is any group.

Fragments $\mathcal{L}_{\Box[\forall]_{i,A}}$ and $\mathcal{L}_{KB_{i,A}}$ restrict the resp. modalities to individuals and the full group A.

Theorem 1

The proof system $\mathbf{\Box[\forall]_I}$ *is sound and complete for* $\mathcal{L}_{\Box[\forall]_I}$, *and the logic* $\mathcal{L}_{\Box[\forall]_I}$ *is decidable. These properties are inherited by the proof system* $\mathbf{\Box[\forall]_{i,A}}$ *and the logic* $\mathcal{L}_{\Box[\forall]_{i,A}}$.

($S4_{\Box}$)	S4 axioms and rules for $\Box_I$		
($S5_{[\forall]}$)	S5 axioms and rules for $[\forall]_I$		
Monotonicity	$\Box_J\varphi \to \Box_I\varphi$,	$[\forall]_J\varphi \to [\forall]_I\varphi$	(for $J \subseteq I$)
Inclusion	$[\forall]_I\varphi \to \Box_I\varphi$		

The logic of group knowledge and group belief (1/2)

Theorem 2

The proof system $\boldsymbol{KB_{i,A}}$ *is sound and complete for* $\mathcal{L}_{KB_{i,A}}$*. Moreover, this logic is decidable.*

	Axioms & rules of normal modal logic for K **&** B
	Stalnaker's Epistemic-Doxastic Axioms:
Truthfulness of knowledge (T)	$\mathbf{K}_\alpha\varphi \to \varphi$
Pos. Intro. of knowledge (KK)	$\mathbf{K}_\alpha\varphi \to \mathbf{K}_\alpha\mathbf{K}_\alpha\varphi$
Consistency of Beliefs (CB)	$\mathbf{B}_\alpha\varphi \to \neg\mathbf{B}_\alpha\neg\varphi$
Strong Pos. Intro. of beliefs (SPI)	$\mathbf{B}_\alpha\varphi \to \mathbf{K}_\alpha\mathbf{B}_\alpha\varphi$
Strong Neg. Intro. of beliefs (SNI)	$\neg\mathbf{B}_\alpha\varphi \to \mathbf{K}_\alpha\neg\mathbf{B}_\alpha\varphi$
Knowledge implies Belief (KB)	$\mathbf{K}_\alpha\varphi \to \mathbf{B}_\alpha\varphi$
Full Belief (FB)	$\mathbf{B}_\alpha\varphi \to \mathbf{B}_\alpha\mathbf{K}_\alpha\varphi$
	where $\alpha \in A \cup \{A\}$.

In addition to the above axioms, we also need the following:

Group Knowledge Axioms:

Super-Introspection (SI)	$\mathbf{B_i}\varphi \rightarrow \mathbf{K_A B_i}\varphi$
Weak Monotonicity (WM)	$(\mathbf{K_i}\varphi \wedge \mathbf{B_A}\varphi) \rightarrow \mathbf{K_A}\varphi$
Consistency of group Belief with Distributed knowledge (CBD)	$(\bigwedge_{\mathbf{i \in A}} \mathbf{K_i}\varphi_{\mathbf{i}}) \rightarrow \langle \mathbf{B_A} \rangle (\bigwedge_{\mathbf{i \in A}} \varphi_{\mathbf{i}})$

for any A-indexed set of formulas $\{\varphi_i \mid i \in A\}$.

The **dynamic language** $\mathcal{L}_{\Box[\forall]_I[\mathsf{share}_I]}$ is defined recursively as

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_I \varphi \mid [\forall]_I \varphi \mid [\mathbf{share}_I]\varphi$$

(where $p \in \mathsf{Prop}$ and $I \subseteq A$ is any group);

$\mathcal{L}_{KB_{i,A}[\mathsf{share}_A]}$ is given by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B_i\varphi \mid B_A\varphi \mid K_i\varphi \mid K_A\varphi \mid [\mathbf{share}_A]\varphi$$

(where $p \in \mathsf{Prop}$, and $i \in A$ is any agent).

We interpret the **share modality** as follows:

$$[\![[\mathsf{share}_I]\varphi]\!]^{\mathfrak{M}} = [\![\varphi]\!]^{\mathfrak{M}(\mathsf{share}_I)}.$$

Theorem 3

The proof systems $\Box[\forall]_I[\mathsf{share}_I]$ and $KB_{i,A}[\mathsf{share}_A]$ are sound and complete for $\mathcal{L}_{\Box[\forall]_I[\mathsf{share}_I]}$ and respectively $\mathcal{L}_{KB_{i,A}[\mathsf{share}_A]}$. Moreover, these logics are provably co-expressive with their static bases $\mathcal{L}_{\Box[\forall]_I}$ and respectively $\mathcal{L}_{KB_{i,A}}$, and thus decidable.

$(\Box[\forall]_I)$	**Axioms and rules of** $\Box[\forall]_I$	
$([\mathsf{share}_I])$	**Axioms and rules of normal modal logic for** $[\mathsf{share}_I]$	
	Reduction Axioms for $[\mathsf{share}_I]$:	
(Atomic Reduction)	$[\mathsf{share}_I]p \leftrightarrow p$	(for atomic propositions p)
(Negation Reduction)	$[\mathsf{share}_I]\neg\varphi \leftrightarrow \neg[\mathsf{share}_I]\varphi$	
($\Box$-Reduction)	$[\mathsf{share}_I]\Box_J\varphi \leftrightarrow \Box_{J/+I}[\mathsf{share}_I]\varphi$	
($\forall$-Reduction)	$[\mathsf{share}_I][\forall]_J\varphi \leftrightarrow [\forall]_{J/+I}[\mathsf{share}_I]\varphi$	where: $J/+I := J \cup I$ when $I \cap J \neq \emptyset$, $J/+I := J$ when $I \cap J = \emptyset$.

Completeness: correspondences

$$
\begin{tikzcd}
\textbf{(Alexandroff) Topo-E-Model} \arrow[r, leftrightarrow, "\text{Equiv. w.r.t. } \mathcal{L}_{\Box[\forall]_I}"'] & \text{Relational Evidence Model} \arrow[d, leftrightarrow] \\
 & \text{Standard Pseudo-Model for } \mathcal{L}_{\Box[\forall]_I} \\
\text{Pseudo-Model for } \mathcal{L}_{\Box[\forall]_I} \arrow[r, "\text{Equiv. w.r.t. } \mathcal{L}_{\Box[\forall]_I}"'] & \text{Associated Model} \arrow[u] \\
\textbf{Pseudo-Model for } \boldsymbol{\mathcal{L}_{KB_{i,A}}} \arrow[r, leftrightarrow, "\text{Equiv. w.r.t. } \mathcal{L}_{KB_{i,A}}"'] & \textbf{Pseudo-Model for } \boldsymbol{\mathcal{L}_{\Box[\forall]_{i,A}}} \arrow[u, "\text{Bisim. w.r.t. } \mathcal{L}_{\Box[\forall]_{i,A}}"']
\end{tikzcd}
$$

1. Topological knowledge
2. The problem
3. Formal definitions
4. Axiomatizations
5. Conclusions

Main contributions

1. Logic $\boldsymbol{KB_{i,A}}$
   - Non-monotonic, evidence-based notions of **(virtual) group knowledge and group belief**
   - Sound, complete, and decidable
   - **Better suited** to epistemic dynamics induced by evidence-sharing than traditional distributed knowledge
   - Small step towards applying topological semantics to **realistic, practical settings**: distributed computing, epistemology of social networks
2. Auxiliary tool: logic $\Box[\forall]_{\boldsymbol{I}}$
   - Sound, complete, and decidable

Future work

1. Axiomatizing $\boldsymbol{KB_I}$: language of group knowledge and belief for arbitrary subgroups I
   - Language is decidable
   - Not clear how to generalize completeness proof from $\boldsymbol{KB_{i,A}}$
2. Optimizing our **model checker** for $\mathcal{L}_{\Box[\forall]_I}$
   - Haskell-based symbolic model checker for spatial logics
   - Can be used for plausibility models
   - Efficiency of implementation can be improved

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Saving Group Monotonicity?

Two previous attempts: enforcing Group Monotonicity.

Two-agent solutions: save $K_i \to K_A$ by...

1. ...restricting individual knowledge (Ramírez, 2015)
   - i.e. individual knowledge depends on the evidence of other agents,
2. ...or expanding group knowledge (Fernández, 2018)
   - by only pooling together (a.k.a. cherry picking) evidence that constitutes knowledge.

But: resulting notions of (group) knowledge have undesirable properties.

1. Ramírez' individual knowledge depends on what other agents know
2. Ramírez' solution does not generalise single-agent case
3. Fernández' group knowledge is unattainable in practice

Open problem: completeness of $\boldsymbol{KB_I}$

Expectation: we can use the approach from the proof for $\boldsymbol{KB_{i,A}}$.

- Extend the proof to all subgroups
- Main correspondence:
  1. Pseudo-models for $\mathcal{L}_{\Box[\forall]_I}$ as pseudo-models for $\mathcal{L}_{KB_I}$
     - Recovering knowledge and belief relations from evidence is easy: only one option
  2. Pseudo-models for $\mathcal{L}_{KB_I}$ as pseudo-models for $\mathcal{L}_{\Box[\forall]_I}$
     - Recovering evidence relations from knowledge and belief is **not easy**: not uniquely determined

2. **Pseudo-models for $\mathcal{L}_{KB_I}$ as pseudo-models for $\mathcal{L}_{\Box[\forall]_I}$**

is a challenge! In particular:

- **Pseudo-models for $\mathcal{L}_{\Box[\forall]_I}$ (evidence)**:
  belief can be expressed in terms of maximal states w.r.t. evidence
- **Pseudo-models for $\mathcal{L}_{KB_I}$ (knowledge)**:
  similar: belief can be expressed in terms of maximal states w.r.t. knowledge
- **To show**: maximal states in terms of soft evidence agree with maximal states in terms of knowledge

Hypothesis: to prove this, we need different definitions for the recovered evidence relations in the case of $\mathcal{L}_{KB_I}$.

Model checking

Symbolic model checking is **efficient** model checking.

- Compact representation of model
- Efficient representation of formula

Model checking φ on $(\mathcal{F}, s)$:

1. Construct **Binary Decision Diagram** (BDD) of $\|\varphi\|_{\mathcal{F}}$;
2. Check whether BDD of φ accepts valuation of s.

We model-check on structures equivalent to topo-e-models.

Definition 4 (Symbolic Topo-Structure)

A **symbolic topo-structure** is a tuple $\mathcal{F} = (\mathsf{Prop}, \theta, E, O)$ s.t.

1. Prop is the *vocabulary* (finite)
2. θ is the *state law* (boolean formula)
3. A *state* is a set $s \subseteq \mathsf{Prop}$ satisfying θ
4. $E = (E_i)_{i \in A}$ is the *evidence*
5. $O = (O_i)_{i \in A}$ are the *observables*
6. Each O_i decides a partition over the states of $\mathcal{F}$.

Soft and hard evidence are represented by propositional variables!

We focus on the semantics of **soft** and **hard** evidence:

Definition 3 (Symbolic Semantics of $\mathcal{L}_{\Box[\forall]_I}$)

Given a nonempty subgroup $I \subseteq A$, we define

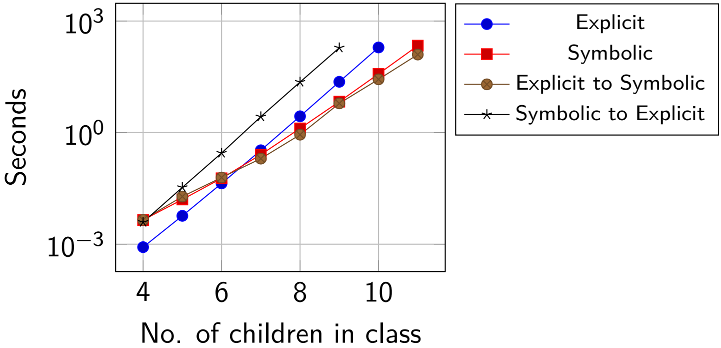
$(\mathcal{F}, s) \vDash \Box_I \varphi$ *iff* *for all states t of $\mathcal{F}$,*
if $s \cap O_I = t \cap O_I$ and $\boldsymbol{s \cap E_I \subseteq t \cap E_I}$,
then $(\mathcal{F}, t) \vDash \varphi$

$(\mathcal{F}, s) \vDash [\forall]_I \varphi$ *iff* *for all states t of $\mathcal{F}$,*
if $s \cap O_I = t \cap O_I$,
then $(\mathcal{F}, t) \vDash \varphi$.

Obtain **boolean translation** $\|\varphi\|_{\mathcal{F}}$ of $\varphi \in \mathcal{L}_{\Box[\forall]_I}$ such that

$$(\mathcal{F}, s) \vDash \varphi \text{ iff } s \vDash \|\varphi\|_{\mathcal{F}}.$$

On a scalable model, we checked a formula closely related to (BDK) *Consistency of Group Belief with Distributed Knowledge*.



Implementation: Boolean translation of $[\forall]_I\varphi$

The boolean translation of a formula $[\forall]_I\varphi$ is

$$\|[\forall]_I\varphi\|_{\mathcal{F}} := \forall(V \setminus O_I)(\theta \to \|\varphi\|_{\mathcal{F}}).$$

It is implemented in Haskell as

```haskell
bddOf stm (Forall ags f) = forallSet otherps $ imp (theta stm) (bddOf stm f) where
  otherps = map fromEnum $ vocab stm \\ evOrObsOfGroup ags (obs stm)
```

Implementation: Boolean translation of $\Box_I\varphi$

The boolean translation of a formula $\Box_I\varphi$ is

$$\|\Box_I\varphi\|_{\mathcal{F}} := \forall E_I' \left(\bigwedge_{e_i \in E_I} (e_i' \leftrightarrow e_i) \rightarrow \forall(\mathrm{Prop} \setminus O_I) \left(\bigwedge_{e_i \in E_I} (e_i' \rightarrow e_i) \wedge \theta \rightarrow \|\varphi\|_{\mathcal{F}} \right) \right).$$

It is implemented in Haskell as

```haskell
bddOf stm (Box ags f) = forallSet evPrime $ imp evAtState evImpliesf
  where
    ev          = map fromEnum $ evOrObsOfGroup ags (evidence stm)
    evPrime     = map fromEnum $ take (length ev) [freshp (vocab stm) ..]
    primeMap    = Data.IntMap.fromList $ zip ev evPrime
    evAtState   = conSet [equ (var $ primeMap Data.IntMap.! e) (var e) | e <- ev]
    stateSatEv  = conSet [imp (var $ primeMap Data.IntMap.! e) (var e) | e <- ev]
    otherps     = map fromEnum $ vocab stm \\ evOrObsOfGroup ags (obs stm)
    evImpliesf  = forallSet otherps $ imp (con stateSatEv (theta stm)) (bddOf stm
          f)
```